

Math & Music

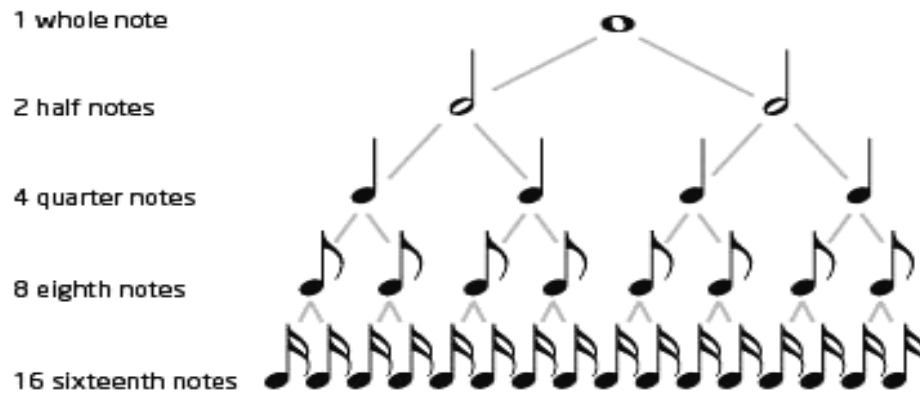
I. Introduction

As a math student also fascinated by music, I have a zest for both disciplines. My passion in math started from primary school, where I was amused by how precise and powerful the math language is. Math is everywhere, and its applications reach into not only academic fields but also our daily life. Everyone in the world has used mathematical operations, for instance when calculating a tip at a restaurant. Math helps me understand the world and solve real-life problems. Music is one way I tune my heart, it helps me tune out the outside world, so I can enjoy my own time, and have a conversation with myself. I didn't realize that how much I care about music until I started to play piano in the March of 2016. As a piano beginner, I became enthralled by the components of a song. To my surprise, the musical theories introduced in class reminded me of mathematical concepts. I decided to explore the correlation between mathematics and music.

II. How musicians use mathematics

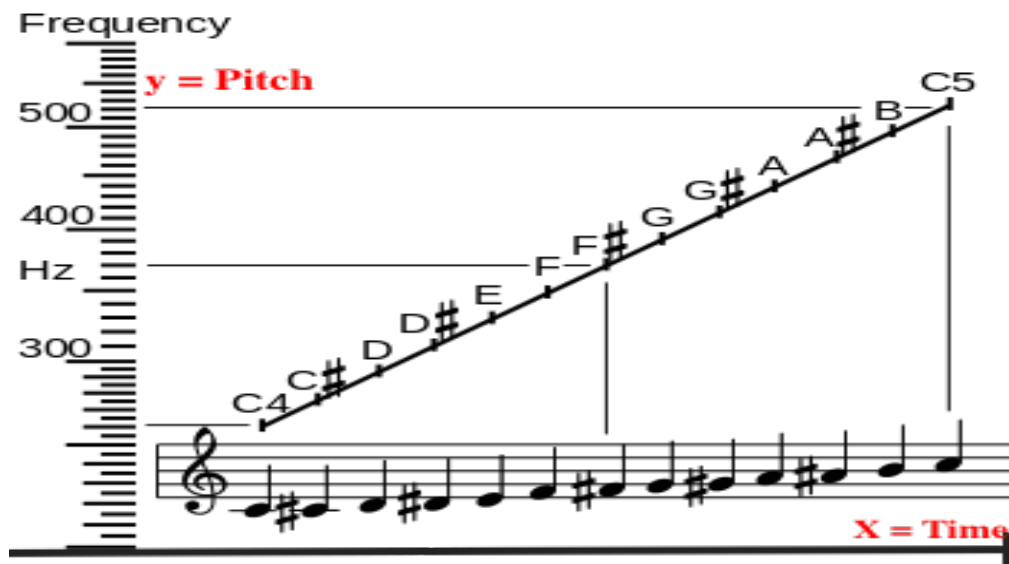
Mathematics can describe many musical phenomena and concepts. The role notes play in music is like the role that played by numbers in math. As we can see from the note tree (Figure 1), a whole note has two times the value as a half note; a half note is twice longer as a quarter note, etc. The time length of these notes follows a geometric sequence where the value of each note after the whole note is equal to the time value of the previous note multiplied by 0.5. Let's suppose the time value of the whole note is k seconds, then the time value of the n th note is equal to $2^{1-n}k$

seconds where n is greater than 1. Therefore, for $n=2$ (i.e. for the half note), we have $2^{-1}k$ seconds; and $n=3$ (the quarter note), we have $2^{-2}k$ seconds; and so on.



(Figure 1: "Note And Rest Values." *Note And Rest Values*. N.p., n.d. Web. 28 Nov. 2016.)

The music notation used in Europe since the eleventh century is similar to graphs of discrete functions in a two-dimensional Cartesian coordinate ^[3]. In Figure 2 below, the y-axis is pitch, which is a discrete value, while the x-axis represents time, which is continuous.



(Figure 2: "Pitch (music)." *Wikipedia*. Wikimedia Foundation, n.d. Web. 28 Nov. 2016.)

In the western system of musical notation, meter is represented by a *time signature*, (a/b), where the lower number (b) stands for the note that gets one beat (4 means a quarter note gets a beat) and the upper number (a) indicates the number of beats in a measure. For example, $3/4$ represents

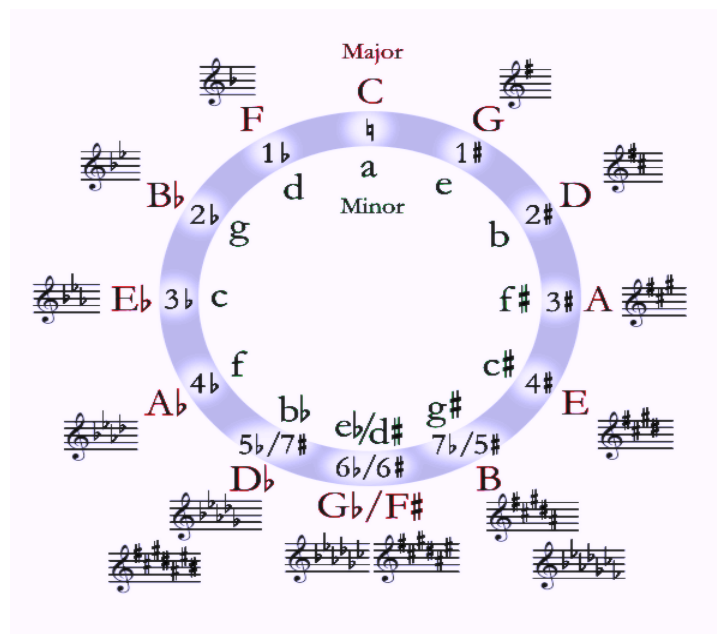
that there are three beats in total in a measure where each quarter note gets one beat.

3/4 Time



(Figure 3: @guitarnoise1. "Time Signatures - Bass for Beginners # 17 - Guitar Noise." *Guitar Noise*. N.p., 2014. Web. 28 Nov. 2016.)

Other than the fundamental music notations, many other musical concepts can be explained by mathematics. The circle of fifths is a diagram that illustrates a key's associated major and minor keys [2]. For any given key, the nearby chords are the associated major and minor key. The outside circle represents the major diatonic scales while the inside circle shows the minor

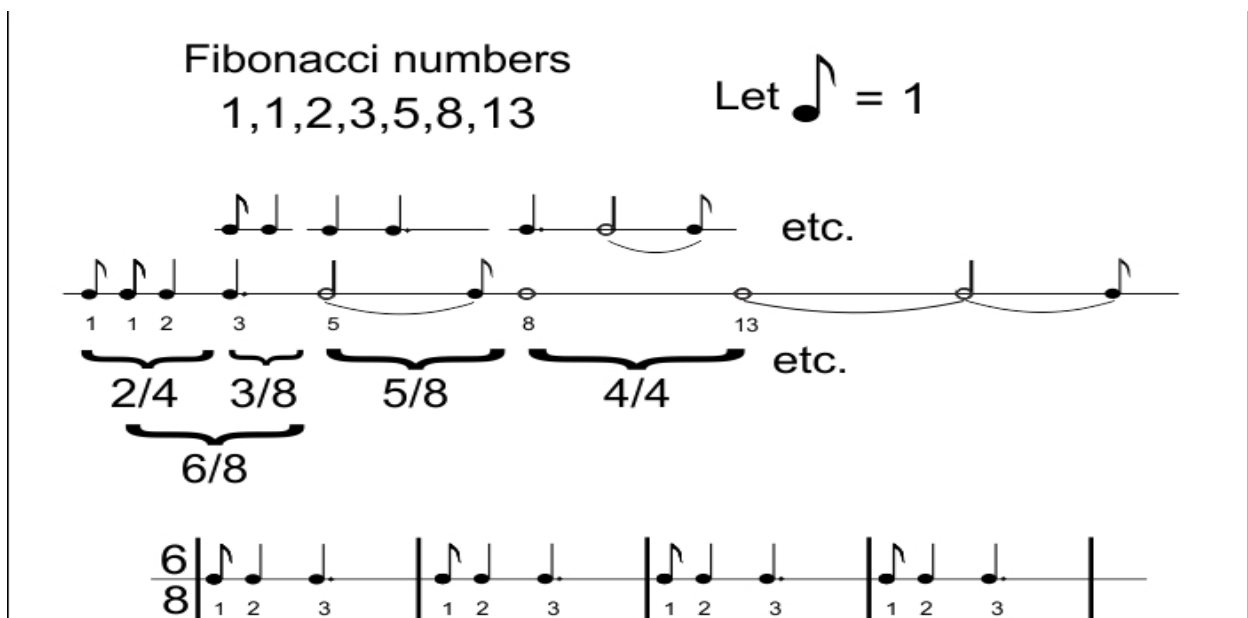


diatonic scales.

(Figure 4: John, By. "John D. Cook." *John D Cook*. N.p., n.d. Web. 28 Nov. 2016.)

What is special about the circle of fifth? The cycles of the fifths covers the chromatic scale while the cycles of other intervals (except the cycles of fourths) partition the chromatic scale into smaller groups of notes. We know that there are 7 semitones in the fifth and the notes on the chromatic scale can be considered as a group of integers modulo 12. The interesting math fact behind it is $7^2 = 49 = 12 * 4 + 1$, the modulo of 12 is 1, which shows that seven half-steps constitutes a perfect fifth.

The Italian mathematician Leonardo Fibonacci developed the Fibonacci sequence that is a series of number starting with one and every successive term is the sum of the two previous numbers. The first seven numbers in the Fibonacci sequence^[5] are 1,1,2,3,5,8,13. The sequence of Fibonacci ratios is the list of numbers obtained from the ratios of each Fibonacci number to its previous Fibonacci number; the first six Fibonacci ratios are 1, 2, 3/2, 5/3, 8/5, and 13/8. Below is one example that shows Fibonacci ratios are related to key frequencies of notes.

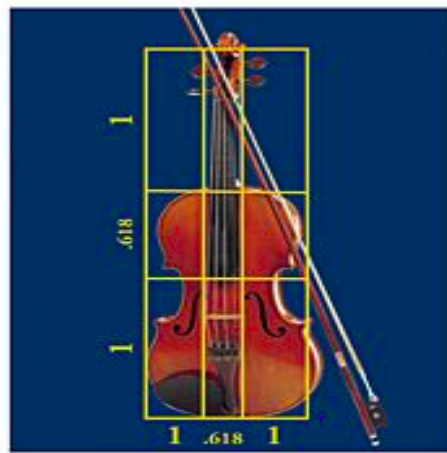


(Figure 5: "Music, The Fibonacci Sequence and Phi." *Order in Chaos*. N.p., 2013. Web. 26 Nov. 2016.)

The dominant note of a major scale is the fifth tone. This is the eighth note of all thirteen notes that comprise the scale. This ratio of eight to thirteen ($8/13 \approx 0.615$) is related to the golden ratio

$$\psi \approx 1.618^{[4]}.$$

Mathematics can also explain how instruments are built and work, for example how strings vibrate at certain frequencies ($f = (\frac{1}{2} L) \sqrt{T/\mu}$). The golden ratio is applied to the design of the violin.



1 : .618

(Figure 6: Ibid)

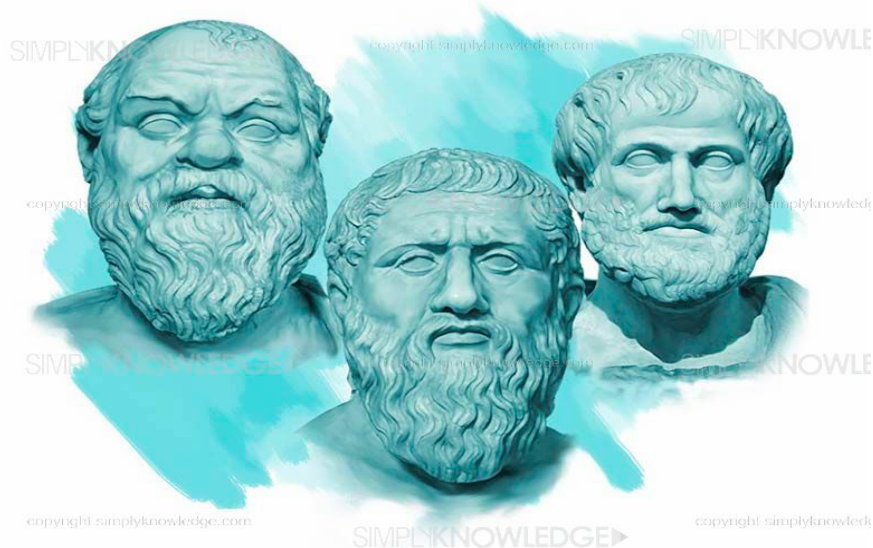
Bach is one of the most famous musicians that have very mathematical compositions. The Baroque era, of which Bach was the master, was very aware of the symbolic significance of numbers concerning religion. It soon became obvious that numerical relationships were significant in Bach's work. For example, Bach's Well-Tempered Clavier (1722) fugues in all 24 major and minor keys. Bach was able to write in every key because at that time mathematicians found better ways to calculate the 12th root of two. Knowing how to get the 12th root of two helped solve the musical problem of dividing the octave into 12 equal intervals involves splitting

sound waves into ratios rather than equal lengths and But before the Baroque time, people don't know modulate so the music always stayed in the same key ^[11].

III. How *mathematicians* play music

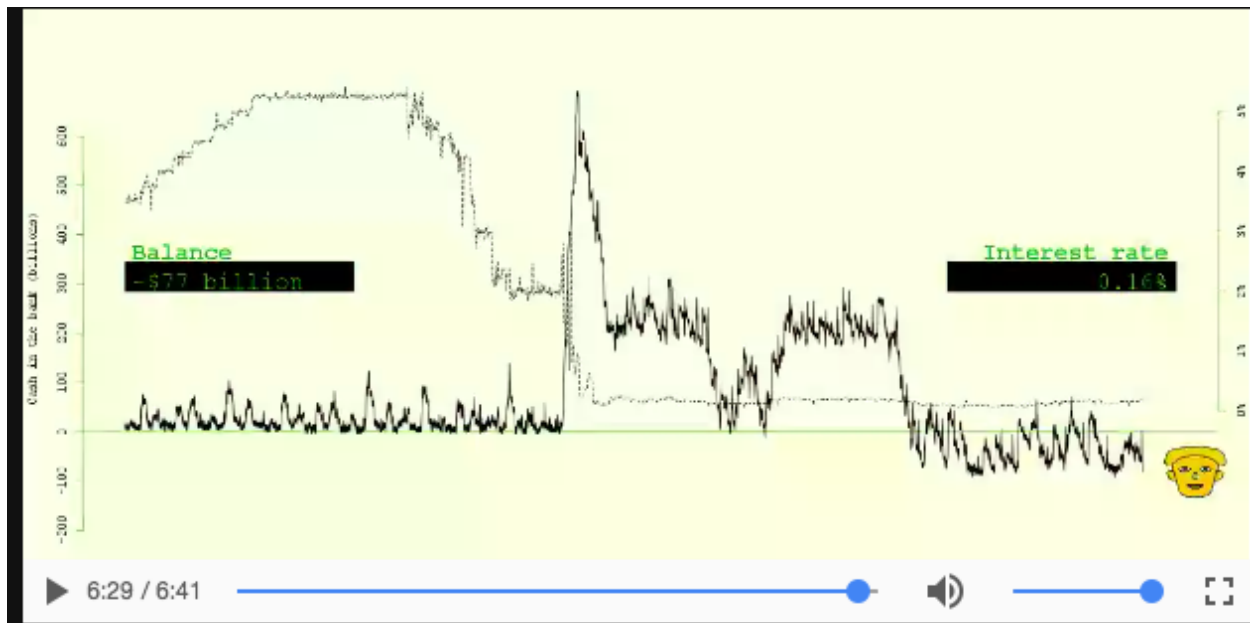
Musicians use mathematics and many mathematicians are also fascinated with music theory since the Ancient Greeks because music theory and composition require an abstract way of thinking and contemplation. This method of thinking is similar to that required for pure mathematical thought.

During 7th century BC, the ancient Greek mathematics education^[1] was divided into four sections: number theory, geometry, music and astronomy. The division of mathematics into four sub-topics is called a quadrivium (mathematical arts). The four-way division of mathematics, which including music should be studied as part of mathematics, lasted until the end of the mid ages (approximately 1500 AD) in European culture. During this period, some prominent mathematicians were also music theorists such as Aristotle, Pythagoras and Plato^[9]; these three Greek scholars had elaborated music rules from mathematical ideas.



(Figure 7: "Plato." *Biography of Plato* | *Simply Knowledge*. N.p., n.d. Web. 28 Nov. 2016.)

Nowadays some mathematicians plot real life data as music, analyze music data for patterns and etc. To help the blind people to visualize the data in their head and to help people get excited about data, some mathematicians plot data in music form. They will first define the rhythm, pitch, tempo and volume based on the data they are using. For example in Figure 8, Thomas Levine (a current popular mathematician) analyzes data about financial crisis^[6]. The left channel is distance to debt ceiling (higher notes means closer to the debt ceiling) and the right channel is interest rate. He defined the chords as daily movements: high/major equals to the money that comes into the market and low/minor is the money that leaves the market.



(Figure 8: "Synthesizers • /r/synthesizers." *Reddit*. N.p., n.d. Web. 27 Nov. 2016.)

Another way to convert data into musical form is to use some sheet music software to use every beat of music as a number from the data table^[10]. I haven't figured out how to do it. But one example Thomas Levine did is displayed as Figure 9.

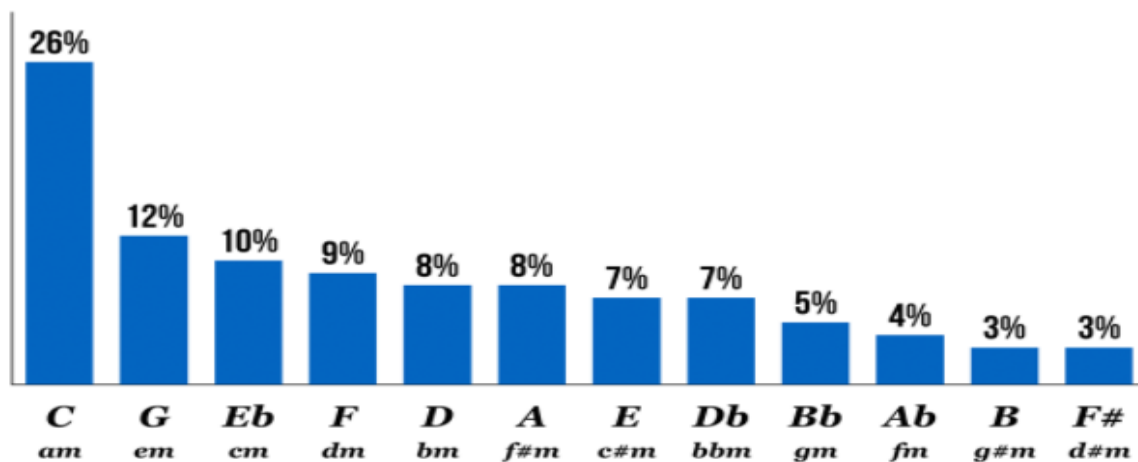
	A	B	C	D	E	F	G	H	I	J	K	L
1	Piano left	Piano right				Melody	Verse1	Verse2	Verse3	Verse4		
17												
18			B4	D4	F#4	B5	in	and	to	bove		
19						B5	a	like	the	the		
20	G2	G3				B5	can	fai	wa	wa		
21												
22			B4	D4	F#4	G5	yon					
23						G5						
24			B4	D4	F#4	G5	ex					
25						B5	ca					
26	G2	G3				D5	vat					
27			B4	D4	F#4							
28						D5	ing					
29			B4	D4	F#4							
30						C5	for					
31						B5	a					
32	D2	D3				A5	mine					
33												
34												
35			D4	E4	G4							
36			D4	E4	G4	A5	Dwelt	her				



(Figure 9: Ibid)

Other than plotting data in a music form, another mathematician, Dave Carlton analyzed music data to help music companies grow their business or help composers understand their customers' reviews about their music. For example, another popular mathematician Dave Carlton analyzed the chords of 1300 songs to find patterns^[7]. After analyzing these data, he answered different interesting questions such as what are most popular keys and most used chords in the 1300 songs? Or more detailed questions like what is the most common chord progression used in these songs and what chords are most likely to come after C major etc. Figure 10 is one example of their findings.

Most popular keys in music



(Figure 10: Carlton, Author Dave. "I Analyzed the Chords of 1300 Popular Songs for Patterns. This Is What I Found." *The Hooktheory Blog*. N.p., n.d. Web. 27 Nov. 2016.)

IV. Conclusion

While the Ancient Greek scholars considered music as mathematical art, I think music and math are two complementary subjects: musicians use mathematical concepts to understand music structures and mathematicians get inspirations from music. The English mathematician James Joseph Sylvester said, “May not music be described as the mathematics of the sense, mathematics as music of the reason? The musician feels mathematics, the mathematician thinks music: music the dream, mathematics the working life”. James’ words precisely describe how mathematicians and musicians think about math and music. The relationship between math and music is so deeply related and immense that it can take one’s whole life to study it. I am glad that I am on the path of learning both math and music.

References:

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- [10] David J. Benson, *Music: A Mathematical Offering*, Cambridge University Press, 2007
- [11] Kozloff, Steve KoppesPhoto by Robert. "Combining Math and Music." *The University of Chicago*. N.p., n.d. Web. 29 Nov. 2016.